UC **SANTA BARBARA**

Curvature Done Optimally

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Introduction

Curvature is a central concept in Riemannian geometry, and bounds on the various curvatures of a manifold M translate into useful constraints on the geometry and topology of M. In particular, lower bounds on the Ricci curvature Ric_M of M play a key role in many important theorems.

We discuss an alternate notion of "curvature bounded below by K" for compact Riemannian manifolds, which only involves the distance on the manifold and the volume measure of the manifold. We will show that in the Riemannian setting, a manifold has Ricci curvature $\geq K$ if and only if it satisfies this alternate condition. This new definition does not explicitly use the Riemannian structure, and thus can be generalized to a broader, nonsmooth class of metric measure spaces.

Comparison Geometry

To get an idea of manifolds with $\mathrm{Ric}_M \geq K$, we can look at the model spaces M^n_K of constant sectional curvature K, where

$$M^n_K = \begin{cases} \text{the sphere } S^n(K), & K > 0 \\ \text{Euclidean space } \mathbb{R}^n, & K = 0 \\ \text{hyperbolic space } \mathbb{H}^n(K), & K < 0. \end{cases}$$

Intuitively, in positive curvature geodesics diverge then converge, in zero curvature they diverge at a constant rate, and in negative curvature they diverge increasingly rapidly.

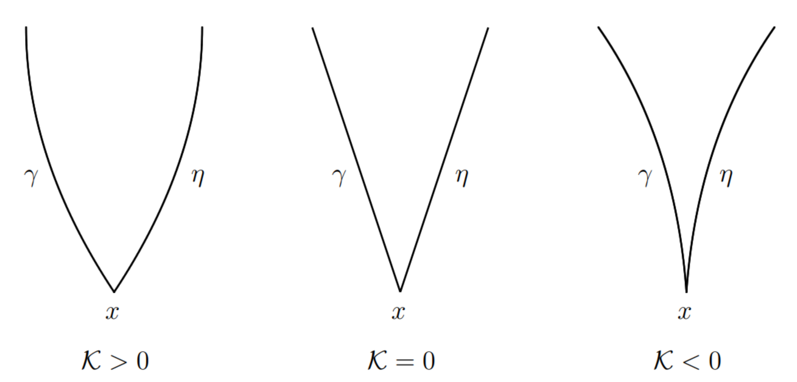

Figure 1. Geodesics in positive, zero, and negative curvature. Image from [3].

Optimal Transport

Optimal transport studies the most efficient way to transport some amount of mass from one configuration to another, such as moving a pile of sand to build a sandcastle:

We can view the configurations of masses as probability measures μ and ν on M and measure efficiency via minimizing the cost

$$\int_M d(x, T(x))^2 \mathrm{d}\mu(x)$$

over maps $T\colon M \to M$ satisfying $T_\sharp \mu = \nu$. McCann showed that if M is compact and $\mu = \rho_0 \,\mathrm{vol}, \nu = \rho_1 \,\mathrm{vol}$, where vol is the normalized volume measure on M, then there exists a unique **optimal transport map** T minimizing the above cost. Moreover, T is of the form

$$T(x) = \exp_x(\nabla\varphi(x))$$

for some semiconvex $\varphi\colon M \to \mathbb{R}$, and the Jacobian determinant of T at x is equal to $\rho_0(x)/\rho_1(T(x))$ μ-almost everywhere.

The Wasserstein 2-Distance

Let $\mathcal{P}_2^{ac}(M)$ be the set of all probability measures on a compact manifold M which are **absolutely continuous** with respect to vol, meaning measures for which we can write $\mu = \rho \,\mathrm{vol}$ for some density ρ. We can give this space a metric by defining

$$W_2(\mu, \nu) = \left(\int_M d(x, T(x))^2 \mathrm{d}\mu(x) \right)^{\frac{1}{2}},$$

where T is the optimal transport map from μ to ν. This distance is called the **2-Wasserstein distance**, and can be defined more generally for metric measure spaces via an alternate formulation of the optimal transport problem.

By the work of McCann, for any two measures $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}$ there is a unique Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$ between them, so that $W_2(\mu_s, \mu_t) = |t-s| W_2(\mu_0, \mu_1)$ for all $0 \leq s, t \leq 1$. Moreover, there must exist $T\colon [0,1] \times M \to M$ given by

$$T(t, x) = \exp_x(t\nabla\varphi(x)),$$

so that for each $t \in [0,1]$, the map $T_t\colon M \to M$ defined by $T_t(x) = T(t, x)$ is the optimal transport map from μ_0 to μ_t. Therefore, we can write $\mu_t = (T_t)_\sharp \mu_0$, and for each t the Jacobian determinant of T_t at x is $\rho_0(x)/\rho_t(T_t(x))$ μ_0-almost everywhere. Observe that by properties of the exponential map, $d(x, T_1(x)) = |\nabla\varphi(x)|$ for all x, hence

$$W_2(\mu_0, \mu_1)^2 = \int_M |\nabla\varphi|^2 \mathrm{d}\mu.$$

Optimal Transport Maps and Ricci Curvature

The key connection between optimal transport on M and the Ricci curvature of M is that Ricci curvature features in a differential inequality for the Jacobian determinant of the map $T(t, x) = \exp_x(t\nabla\varphi(x))$.

Let $J_t(x)$ be the Jacobian of T_t at x, and let $\mathcal{J}_t(x) = \det J_t(x)$. We can write $\mathcal{J}_t(x)$ in terms of Jacobi fields along the geodesic $\gamma(t) = \exp_x(t\nabla\varphi(x))$. Then, via the Jacobi equation and the Cauchy-Schwarz inequality, we obtain the inequality

$$\frac{\mathcal{J}_t''}{\mathcal{J}_t} - \left(\frac{\mathcal{J}_t'}{\mathcal{J}_t}\right)^2 \leq \frac{\mathcal{J}_t''}{\mathcal{J}_t} - \left(1 - \frac{1}{n}\right)\left(\frac{\mathcal{J}_t'}{\mathcal{J}_t}\right)^2 \leq -\mathrm{Ric}(\nabla\varphi, \nabla\varphi).$$

Curvature Bounded Below by K

For $\mu \in \mathcal{P}_2^{ac}(M)$, define the **entropy** of μ by

$$H(\mu) = \int_M \rho \log \rho \,\mathrm{d}\,\mathrm{vol},$$

where $\mu = \rho \,\mathrm{vol}$. We say M has **curvature bounded below by K** if for any measures $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(M)$, the unique Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$ satisfies

$$H(\mu_t) \leq (1-t)H(\mu_0) + tH(\mu_1) - K\frac{t(1-t)}{2} W_2(\mu_0, \mu_1)^2$$

for all $0 \leq t \leq 1$.

References

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[3] Shin-ichi Ohta. Ricci curvature, entropy and optimal transport. 2014.

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The Main Theorem

Theorem 1 (Equivalence of $\mathrm{Ric}_M \geq K$ ***and Curvature Bounded Below By*** K***).*** *A compact Riemannian manifold* M^n *satisfies* $\mathrm{Ric}_M \geq K$ *if and only if it has curvature bounded below by* K.

Proof. First, suppose $\mathrm{Ric}_M \geq K$. Let $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(M)$ be arbitrary, and let $T(t, x) = \exp_x(t\nabla\varphi(x))$ the map associated with the unique Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$. Then we have $\rho_t(T_t(x)) = \rho_0(x)/\mathcal{J}_t(x)$, and so by this change of variables we have

$$\frac{d}{dt}H(\mu_t) = \frac{d}{dt}\int_M \frac{\rho_0}{\mathcal{J}_t}\log\frac{\rho_0}{\mathcal{J}_t}\mathcal{J}_t \mathrm{d}\,\mathrm{vol} = -\int_M \frac{\mathcal{J}_t'}{\mathcal{J}_t}\rho_0 \mathrm{d}\,\mathrm{vol},$$

$$\frac{d^2}{dt^2}H(\mu_t) = -\int_M \left(\frac{\mathcal{J}_t''}{\mathcal{J}_t} - \left(\frac{\mathcal{J}_t'}{\mathcal{J}_t}\right)^2\right)\rho_0 \mathrm{d}\,\mathrm{vol}.$$

By the differential inequality for $\mathcal{J}$, we have

$$\frac{d^2}{dt^2}H(\mu_t) \geq \int_M \mathrm{Ric}(\nabla\varphi, \nabla\varphi)\mathrm{d}\,\mathrm{vol} \geq K\int_M |\nabla\varphi|^2 \mathrm{d}\,\mathrm{vol} = KW_2(\mu_0, \mu_1)^2.$$

Defining $f(t) = H(\mu_t) + K\frac{t(1-t)}{2}W_2(\mu_0, \mu_1)^2$, we see that $\frac{d^2}{dt^2}f(t) = \frac{d^2}{dt^2}H(\mu_t) - KW_2(\mu_0, \mu_1)^2 \geq 0$. Therefore f is convex, and for any $t \in [0,1]$ we have

$$H(\mu_t) + K\frac{t(1-t)}{2}W_2(\mu_0, \mu_1)^2 = f(t) \leq (1-t)f(0) + tf(1) = (1-t)H(\mu_0) + tH(\mu_1),$$

as desired.

The other direction is more complicated, so we only sketch an outline here. A more detailed account can be found in [5]. Suppose M has curvature bounded below by K. Let $x \in M$, $v \in T_xM$ be arbitrary. Take μ_0 to be the normalized volume measure on a small ball $B_\varepsilon(x)$, and take the transport map to be $T_t(x) = \exp_x(t\delta\nabla\varphi(x))$ for some small δ and suitable φ satisfying $\nabla(\varphi(x_0)) = v$. From this we obtain a Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$ given by $\mu_t = (T_t)_\sharp \mu_0$, and choosing ε, δ, and φ carefully, the inequality for $H(\mu_t)$ gives us the desired Ricci curvature bound $\mathrm{Ric}(v, v) \geq K|v|^2$. □

Brunn-Minkowski

One geometric consequence of curvature $\geq K$ is the following Brunn-Minkowski inequality, generalizing the classic Brunn-Minkowski inequality in $\mathbb{R}^n$.

Theorem 2 (Generalized Brunn-Minkowski). *Suppose that* M *has curvature bounded below by* K*. For nonempty, compact* $A_0, A_1 \subseteq M$ *and* $t \in (0,1)$*, let* A_t *be the set of all points* $\gamma(t)$*, where* γ *runs over all unit-length geodesics with* $\gamma(0) \in A_0, \gamma(1) \in A_1$*. Then*

$$\ln \mathrm{Vol}(A_t) \geq (1-t)\ln \mathrm{Vol}(A_0) + t\ln \mathrm{Vol}(A_1) + K\frac{t(1-t)}{2}d(A_0, A_1)^2.$$

The following diagram depicts this inequality in the case $K > 0$, reflecting the fact that geodesics spread out and then return back together in positive curvature.

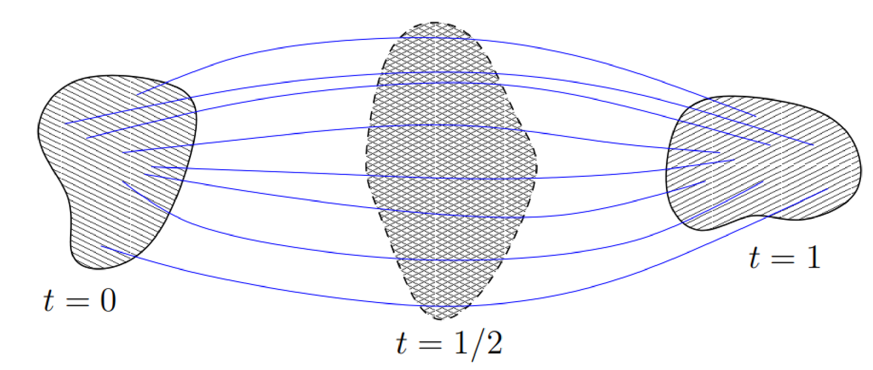

Figure 2. Brunn-Minkowski when $K > 0$. Image from [5].

This inequality can be improved if we introduce a $CD(K, N)$ condition, which encapsulates both "curvature bounded below by K" and "dimension bounded above by N", and in fact the validity of this improved inequality is equivalent to the $CD(K, N)$ condition.